Fault–Tolerant High–Performance Matrix Multiplication: Theory and Practice*

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Abstract

In this paper, we extend the theory and practice regarding algorithmic fault-tolerant matrix-matrix multiplication, $C = AB$, in a number of ways. First, we propose low-overhead methods for detecting errors introduced not only in C but also in A and/or B. Second, we show that, theoretically, these methods will detect all errors as long as only one entry is corrupted. Third, we propose a low-overhead roll-back approach to correct errors once detected. Finally, we give a high-performance implementation of matrix-matrix multiplication that incorporates these error detection and correction methods. Empirical results demonstrate that these methods work well in practice while imposing an acceptable level of overhead relative to high-performance implementations without fault-tolerance.

1 Introduction

The high-performance implementation of many linear algebra operations depends on the ability to cast most of the computation in terms of matrix-matrix multiplication [2, 3, 6, 12]. High-performance for matrix-matrix multiplication itself results from the fact that the cost of moving $b \times b$ blocks of the operands between the layers of the memory hierarchy is proportional to b^2, while this cost can be amortized over $O(b^3)$ computations. These observations impact algorithmic fault-tolerance for linear algebra routines that spend most of their time in matrix-matrix multiplication in the following sense:

- If the matrix-matrix multiplication kernel used is fault-tolerant, the entire operation is largely fault-tolerant.
- Ensuring the integrity of a $b \times b$ block of a matrix can be expected to cost $O(b^2)$ time. This expense can be amortized over the $O(b^3)$ operations performed with that data.

Thus, not only is the availability of a fault-tolerant matrix-matrix multiplication an important first step towards creating fault-tolerant linear algebra libraries, but there is an inherent opportunity for adding fault-tolerance to matrix-matrix multiplication while retaining high-performance.

The primary goal for our mechanism is to detect a maximal fraction of errors while introducing minimal overhead. As argued in the previous paragraph, for the matrix product, with a cubic cost in floating–point arithmetic operations, we can expect to pay at least a quadratic cost. Thus, the goal is to find a mechanism with a quadratic cost. We follow, in

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that sense, the technique described in [13, 14]. In those papers, the correctness of $C = AB$ is established by checking $d = Cw - ABw$ for a checksum vector w. The matrix-matrix multiplication is assumed to have been successful if d is of the order of the errors that could be introduced due to the use of finite precision arithmetic (round-off errors). This is a simple application of Result-Checking [15]. In this paper, we generalize this method to $C \leftarrow \alpha AB + \beta C$, the form of matrix-matrix multiplication that is part of the level 3 Basic Linear Algebra Subprograms (BLAS) [5], and sharpen the theory behind the method. In particular, we show that to *guarantee* detection of a single error introduced in one of the matrices A, B, or C, one must check both $d = Cw - ABw$ and $e = v^T C - v^T AB$ for checksum vectors v and w. Finally, we show how to incorporate the techniques in a high-performance implementation of matrix-matrix multiplication.

The methods we present are closely related to those described in [11]. That paper proposes to augment matrices A, B, and C as

$$A^\star = \left(\frac{A}{v^T A}\right), B^\star = \left(\begin{array}{c|c} B & Bw \end{array}\right), \text{ and}$$
$$C^\star = \left(\frac{C}{v^T C}\,\middle|\,\frac{Cw}{v^T Cw}\right).$$

(Here, both v^T and w are checksum vectors.) By noting that in the absence of errors

$$C^\star = \left(\frac{C}{v^T C}\,\middle|\,\frac{Cw}{v^T Cw}\right) = \left(\frac{A}{v^T A}\right)\left(\begin{array}{c|c} B & Bw \end{array}\right)$$
$$= \left(\frac{AB}{v^T AB}\,\middle|\,\frac{ABw}{v^T ABw}\right) = A^\star B^\star,$$

they show how a comparison of $v^T C$ with $v^T AB$ and Cw with ABw can detect and correct errors introduced in matrix C.

On the surface, when comparing our methods to [11], it may appear that, from an implementation point of view, we simply perform the matrix multiplies separately rather than as part of augmented matrices. (This shows the simple connection between Result-Checking and the approach in [11].) However, our approach differs in a number of ways. First, we go well beyond the approach in [11] by also developing a sound theory behind the detection of errors introduced in A and B. Second, by adopting the techniques developed in [14] we explicitly deal with the question of how to differentiate errors due to corruption from errors due to round-off. Third, we take a very different approach to the correction of detected errors by using a roll-back method. Finally, by adding fault-tolerance to a *high-performance* implementation of matrix-matrix multiplication we verify that the theoretical results can be implemented without sacrificing high performance.

The rest of the paper is structured as follows. In Section 2 we briefly describe the intended domain of application for our methods. In Section 3 we expound upon our theory concerning the effects of the introduction of one error in one of the matrices during a matrix-matrix multiplication. In Section 4 we describe how to take the results from Section 3 from theory to practice (although still at a high level of abstraction). A working fault-tolerant implementation of the matrix product based on a high-performance matrix-matrix multiplication implementation (ITXGEMM [8, 9]) is subsequently given in Section 5. The experimental results in Section 6 reveal the low overhead introduced in the matrix product by our fault–detection mechanism. We briefly discuss the current status of the project in Section 7 and concluding remarks are given in Section 8.

2 Target Application

Within NASA's High Performance Computing and Communications Program, the Remote Exploration and Experimentation (REE) project [1] at the Jet Propulsion Laboratory aims to enable a new type of scientific investigation by taking commercial supercomputing technology into space. Transferring such computational power to space will enable highly-autonomous, flexible missions with substantial onboard analysis capability, mitigating control latency issues due to fundamental light-time delays, as well as inevitable bandwidth limitations in the link between spacecraft and ground stations. To do this, REE does not intend to develop a new computational platform, but rather to define and demonstrate a process for rapidly transferring commercial high-performance computing technology into ultra-low power, fault-tolerant architectures for space.

The traditional method for protecting spacecraft components against faults caused by natural galactic cosmic rays and energetic protons has been radiation-hardening. However, radiation-hardening lowers the clock speed and may increase the required power of a component. Even worse, the time needed to design and bring a radiation-hardened component into production guarantees that it will be outdated when it is ready for use in space. Furthermore, the design and production expenses must be spread over a small number of customers, making the per unit cost very high. Typically, at any given time, radiation-hardened components have a power:performance ratio that is an order of magnitude lower, and a cost that is several orders of magnitude higher than contemporary commodity off-the-shelf (COTS) components. The REE project is therefore attempting to use COTS components in space and handling, via software, the faults that will occur.

Most of the transient faults encountered due to radiation in space will be single event effects (SEEs); their presence requires that the applications be self-checking, or tolerant of

errors, as the first layer of fault-tolerance. Additional software layers will protect against errors that are not caught by the application [4]. For example, one such layer would automatically restart programs which have crashed or hung. This works in conjunction with self-checking routines: if an error is detected, and the computation does not yield correct results after a set number of retries, the error handling scheme aborts the program so that it can be automatically restarted.

In an REE system, there will be many places in which SEEs can cause errors. Layers of memory which are off-processor (main memory, L2 and L3 caches) can be made error detecting and correcting, so that faults to these layers of memory will largely be screened. Most faults will therefore impact the microprocessor and its registers or its L1 cache. SEEs affecting data are particularly troublesome because they typically have fewer obvious consequences than an SEE that impacts code (e.g. in the L1 instruction cache) — the latter would be expected to cause an exception. For this reason, this paper focuses on data corruption, specifically in those components which are on-processor (L1 cache, registers) which cannot be protected through hardware, because while the REE Project can modify the off-processor memory system at relatively low cost, it cannot modify the processor.

A single error correction, double error detection (SECDED) Hamming code will be used to protect off-processor data. This can be applied to cache lines of data when they are written or checked when they are read. In L2 and L3 cache, if the fault rate is sufficiently high, it is possible to scrub data by using a background process on the processor to invalidate lines of data that have not been used in some period of time, though the REE Project does not believe this will be necessary. In main memory, scrubbing will be required. This can either be implemented as a background process (similar to that used for L2 and L3 cache), or in hardware, as an Field Programmable Gate Array (FPGA) or Application Specific Integrated Circuit (ASIC) that is on the memory bus. The rate of scrubbing can be tied to the the error detection rate in order to keep the error rate roughly constant, for power efficiency.

Due to the nature of most scientific codes, including the data processing applications currently being studied by REE, much of their time is spent in certain common numerical subroutines — as much as 70% in one NGST (Next Generation Space Telescope, the planned successor to the Hubble Space Telescope) application, for example. Protecting these subroutines from faults provides one level of protection in an overall software-implemented fault-tolerance scheme.

3 Detecting Errors

In this section we develop a theoretical foundation for error detection in the operation $C = AB$ where C, A, and B are $m \times n$, $m \times k$, and $k \times n$, respectively. Here, we use partitionings of A and B by columns and rows, respectively:

$$A = \left(\begin{array}{c|c|c} a_1 & \cdots & a_k \end{array} \right) \quad \text{and} \quad B = \left(\begin{array}{c} \hat{b}_1^T \\ \hline \vdots \\ \hline \hat{b}_k^T \end{array} \right).$$

We also use two (possibly different) checksum vectors:

$$w = \left(\begin{array}{c} \omega_1 \\ \hline \vdots \\ \hline \omega_n \end{array} \right) \quad \text{and} \quad v^T = \left(\begin{array}{c|c|c} \nu_1 & \cdots & \nu_m \end{array} \right).$$

For simplicity, we first assume that exact arithmetic is employed and then we discuss the tolerance threshold for the case where round-off errors are present.

3.1 Exact arithmetic

Consider the operation $C = AB$ and let $\tilde{C}$ be the matrix computed when at most one element of any one of the three matrices is corrupted during the computation. (We primarily consider a single corruption since most errors will be SEEs.) In other words, view the operation as atomic and assume that before the computation one element of A or B is corrupted or after $C = AB$ has been formed one element of C is corrupted. We can think of the error as a matrix of the form $\eta e_i e_j^T$ added to one of the three matrices; here η is the magnitude of the error and e_k denotes the k–th column of the identity matrix. The possible computed results are then given in Table 1 in the row labeled "$\tilde{C}$". Naturally, we wish to detect the instances in which $F = \tilde{C} - C$ is nonzero (or, in the presence of round-off error, "significant"). Thus, we must compute or approximate the magnitude of F, e.g., as $\|F\|_\infty$, but we must do so without being able to form F. Moreover, relative to the cost of computing C, the computation of the estimation of $\|F\|_\infty$ must be cheap.

Right-sided error detection criterion

Consider now the computation of $d = \tilde{C}w - Cw$, where w is a vector with entries $\omega_i = 1$, $i = 1, \ldots, n$. From Table 1 we see that *if* the corruption is in matrix B or C, $\|d\|_\infty = \|F\|_\infty$. As we do not have C, but a possibly corrupted approximation, $\tilde{C}$, we use $A(Bw)$ instead of Cw in the computation of d; only three matrix-vector multiplications are then required to compute d. These matrix-vector multiplications are cheap relative to a matrix-matrix multiplication. Computing d and its norm is exactly the procedure suggested in [14].

	Matrix Corrupted		
	$\tilde{A} = A + \eta e_i e_j^T$	$\tilde{B} = B + \eta e_i e_j^T$	$\tilde{C} = C + \eta e_i e_j^T$
$\tilde{C}$	$\tilde{A}B$	$A\tilde{B}$	$AB + \eta e_i e_j^T$
$F = \tilde{C} - C$	$\eta e_i \hat{b}_j^T$	$\eta a_i e_j^T$	$\eta e_i e_j^T$
$\|F\|_\infty$	$\|\eta\| \|\hat{b}_j^T\|_\infty$	$\|\eta\| \|a_i\|_\infty$	$\|\eta\|$
$d = Fw$	$\eta e_i \hat{b}_j^T w$	$\eta \omega_j a_i$	$\eta \omega_j e_i$
$\|d\|_\infty$	$\|\eta\| \|\hat{b}_j^T w\|$	$\|\eta\| \|\omega_j\| \|a_i\|_\infty$	$\|\eta\| \|\omega_j\|$
$e^T = v^T F$	$\eta \nu_i \hat{b}_j^T$	$\eta v^T a_i e_j^T$	$\eta \nu_i e_j^T$
$\|e^T\|_\infty$	$\|\eta\| \|\nu_i\| \|\hat{b}_j^T\|_\infty$	$\|\eta\| \|v^T a_i\|$	$\|\eta\| \|\nu_i\|$
criterion	$\|e^T\|_\infty (= \|\nu_i\| \|F\|_\infty)$	$\|d\|_\infty (= \|\omega_j\| \|F\|_\infty)$	$\|d\|_\infty (= \|\omega_j\| \|F\|_\infty)$ or $\|e^T\|_\infty (= \|\nu_i\| \|F\|_\infty)$

Table 1. Some measurements and error detection criteria.

However, if the corruption occurs in A, $\|d\|_\infty = |\eta| |\hat{b}_j^T w|$, which can be small even if $\|F\|_\infty$ is large. In particular, if the elements of the j–th row of B sum to zero, $\|d\|_\infty = 0$ regardless of the the magnitude $\|F\|_\infty$. While this is not likely to happen in practice, the method is clearly not bulletproof for detecting corruption in A. A simple example of a matrix encountered in practice which has rows and/or columns with entries that sum to zero is the matrix derived from a discretization of Poisson's equation using a five-point stencil.

We will refer to the error detection criterion which places checksum vector w on the right as a *right-sided* error detection criterion. This criterion is guaranteed to detect a single error introduced in B or C. It is highly likely to detect such an error introduced in A.

Left-sided error detection criterion

Next, consider the computation $e = v^T \tilde{C} - v^T C$ where v is a vector with entries $\nu_i = 1, i = 1, \ldots, m$. From Table 1 we see that *if* the corruption is in matrix A or C, $\|e^T\|_\infty = \|F\|_\infty$. Again, by computing $v^T C = (v^T A)B$ we can obtain e with only three matrix-vector multiplications. In this case, if the corruption was in B, $\|e^T\|_\infty = |\eta| |v^T a_i|$, which can be small even if $\|F\|_\infty$ is large. In particular, if the elements of the i–th column of A sum to zero, $\|e\|_\infty = 0$. Thus, the method is clearly not completely foolproof for detecting corruption of B.

We will refer to the error detection criterion which places checksum vector v on the left as a *left-sided* error detection criterion. This criterion is guaranteed to detect a single error introduced in A or C. It is highly likely to detect such an error introduced in B.

Two-sided error detection criterion

Clearly, in order to guarantee the detection of the corruption of a single element in one of the three matrices, one must compute $\|d\|_\infty$ if the error is in either B or C, and $\|e\|_\infty$ if the error is in either A or C.

3.2 Tolerance threshold and round-off errors

Unfortunately, computers are not equipped to deal with infinite precision arithmetic and rounding errors due to finite precision arithmetic will occur. In our error detection setting this means that, even if no error is introduced in any of the matrices, it may well be the case that $\|\tilde{C} - C\| \neq 0$.

Round-off error analysis of matrix operations has been a classic area of numerical analysis for the last half-century. A result found in standard textbooks (e.g., [7]) is that for an implementation of the matrix product $C = AB$, based on *gaxpy*, *dot product*, or *outer product* computations, the computed results, $\mathrm{fl}(AB)$, satisfies

$$\|\mathrm{fl}(AB) - AB\|_\infty \leq \max(m, n, k)\ \mathbf{u}\ \|A\|_\infty \|B\|_\infty + O(\mathbf{u}^2),$$

where $\mathbf{u}$ is the unit round-off of the machine (the difference between 1 and the next larger floating point number representable in that machine).

Therefore, our error detection mechanism should declare that an error has occurred when

$$\|d\|_\infty > \tau\ \|A\|_\infty \|B\|_\infty \quad \text{or} \quad \|e^T\|_\infty > \tau\ \|A\|_\infty \|B\|_\infty,$$

where $\tau = \max(m, n, k)\ \mathbf{u}$.

These results on thresholds for detecting errors merely reiterate the observations made in [14].

3.3 Specialization to our situation

As mentioned in Section 2, our primary concern involves a corruption affecting data which reside in the L1 cache. Thus this corruption does not necessarily persist during the entire matrix-matrix multiplication. Therefore, it may be more informative to view matrices C, A, and B partitioned as follows:

$$C = \left(\begin{array}{c|c|c} C_{11} & \cdots & C_{1N} \\ \hline \vdots & \ddots & \vdots \\ \hline C_{M1} & \cdots & C_{MN} \end{array}\right), \quad A = \left(\begin{array}{c|c|c} A_{11} & \cdots & A_{1K} \\ \hline \vdots & \ddots & \vdots \\ \hline A_{M1} & \cdots & A_{MK} \end{array}\right), \quad \text{and} \quad B = \left(\begin{array}{c|c|c} B_{11} & \cdots & B_{1N} \\ \hline \vdots & \ddots & \vdots \\ \hline B_{K1} & \cdots & B_{KN} \end{array}\right),$$

where C_{ij} is $m_i \times n_j$, A_{ip} is $m_i \times k_p$, and B_{pj} is $k_p \times n_j$.

Now C_{ij} is computed as a sequence of smaller updates $C_{ij} \leftarrow A_{ip}B_{pj} + C_{ij}$ and the corruption will be encountered in exactly one such update. In other words, for one tuple of indices (i, j, p) one of the operands is corrupted by changing one element. Let us assume that B_{pj} is corrupted by $\eta e_r e_s^T$. Then the computed matrix $\tilde{C}$ is equal to C except in the (i, j) block, which equals $C_{ij} + \eta a_r^{(i,p)} e_s^T$, where $a_r^{(i,p)}$ denotes the r–th column of A_{ip}. If w again equals the vector of all ones, $\|\tilde{C} - C\|_\infty = |\eta| \|a_r^{(i,p)}\|_\infty$ and $\|\tilde{C}w - Cw\|_\infty = |\eta| \|a_r^{(i,p)}\|_\infty$. It follows that the right-sided detection criterion for detecting errors in B or C still works. The theory behind the left-sided and two-sided detection criteria can be extended similarly.

4 Towards a Practical Implementation

In this section we deal with two issues concerning the practical implementation of a fault-tolerant high-performance matrix-matrix multiplication kernel. First, in addition to error detection, we must also be able to correct any errors that are exposed. Second, in order to maintain high-performance, we must let the theory guide us to a scheme that imposes as little overhead as is possible.

Consider $C = \alpha AB + \beta C$ where C, A, and B have dimensions $m \times n$, $m \times k$ and $k \times n$, respectively. The cost of this operation is $2mnk$ floating point operations (FLOPs).

4.1 Right-sided error detection method

A simple approach is to compute $D = AB$, and check the computed $\tilde{D}$ by determining whether

$$\|\tilde{D}w - A(Bw)\|_\infty < \tau \, \|A\|_\infty \|B\|_\infty.$$

If the condition is met, then $C \leftarrow \alpha D + \beta C$ is performed; otherwise D is recomputed. (Note: our assumption is that a *copy* of A or B is corrupted in some level of cache memory. Thus, the recomputation can use the original data in A and B.) If a more stringent threshold is used, a false error due to round-off can occur. In this case one can determine whether or not $\|\tilde{D}w - A(Bw)\|_\infty$ is exactly equal twice in a row. It it is, C is updated since this would indicate that the scheme resulted in a false detection due to round-off error.

The overhead from error detection is $2mn$ FLOPs for forming $\tilde{D}w$ and $2kn + 2mk$ FLOPs for forming $A(Bw)$ for a total of $2mn + 2kn + 2mk$ FLOPs. In addition, the computations of $\|A\|_\infty$ and $\|B\|_\infty$ cost $O(mk)$ and $O(kn)$, respectively. If even a single error is detected, the cost of the operation doubles. Also, storage for D, mn floating point numbers, is required.

4.2 Left-sided error detection method

A simple approach is to again compute $D = AB$, and check the computed $\tilde{D}$ by testing if $\|v^T\tilde{D} - (v^TA)B\|_\infty < \tau \, \|A\|_\infty \|B\|_\infty$. If the condition is met, then $C \leftarrow \alpha D + \beta C$; otherwise D is recomputed. If $\|v^T\tilde{D} - (v^TA)B\|_\infty$ is exactly equal twice in a row, C is updated since it is assumed that a corruption was erroneously detected.

A more sophisticated approach partitions B, C, and D as

$$B = \left(\begin{array}{c|c|c} B_1 & \cdots & B_N \end{array}\right), \quad (1)$$

$$C = \left(\begin{array}{c|c|c} C_1 & \cdots & C_N \end{array}\right), \text{ and} \quad (2)$$

$$D = \left(\begin{array}{c|c|c} D_1 & \cdots & D_N \end{array}\right), \quad (3)$$

and computes $D_j = AB_j$. After each such computation, the magnitude of $\|v^T\tilde{D}_i - y^TB_i\|_\infty$ is checked, where $y^T = v^TA$ can be computed once and reused. As before, if no error is detected, $C_i \leftarrow \alpha\tilde{D}_i + \beta C_i$; otherwise D_i is recomputed. Now only workspace for one D_i is required and fewer computations need to be repeated when an error is detected. Note that this is not possible for the right-sided approach since for each B_iw the product $A(B_iw)$ must be computed, which is expensive when B_i has few columns, as it is in our implementation (described in the experimental section).

Given a column partitioning of matrices D_j and B_j of width b, the overhead from error detection is now $2mk$ FLOPs for forming $y^T = v^TA$, $2mb$ FLOPs for forming $v^T\tilde{D}_j$ and $2kb$ FLOPs for forming v^TB_j. Taking into

account that n/b panels of D must be computed, the total overhead becomes $2mn + 2kn + 2mk$ FLOPs, equivalent to the cost of the right-sided error detection scheme above. In addition, the computations of $||A||_\infty$ and $||B_j||_\infty$, $j = 1, \ldots, N$, cost $O(mk)$ and $O(kn)$, respectively. If a single error is detected during the update of C, only $2mbk$ FLOPs are repeated. In this case, only storage for one panel D_j, mb floating point numbers, is required.

4.3 Two-sided error detection

Naturally the two above-mentioned techniques can be combined to yield a two-sided error detection method. Here all of D is computed using a left-sided error detection method, after which a right-sided error detection method is used to verify that no undetected errors slipped by. If no errors are detected, C is appropriately updated.

The computational cost of two-sided error detection is exactly twice that of the one-sided error detection methods. Storage for all of D, or mn floating point numbers, is required. However, the left-sided error detection scheme will almost always detect errors and thus the overhead for correcting a single error is only $2mbk$ FLOPs.

4.4 Reducing overhead

Even in the case where no error is detected, the above schemes, particularly the right- and two-sided approaches, carry a considerable overhead in required workspace. In addition, if an error is detected with these methods, the cost of recomputation can double the overall cost of the matrix-matrix multiplication. In this section we discuss how both of these overheads can be overcome.

Specifically, partition C, A, and B as

$$C = \left(\begin{array}{c|c|c} C_{11} & \cdots & C_{1N} \\ \hline \vdots & \ddots & \vdots \\ \hline C_{M1} & \cdots & C_{MN} \end{array}\right), \quad (4)$$

$$A = \left(\begin{array}{c|c|c} A_{11} & \cdots & A_{1K} \\ \hline \vdots & \ddots & \vdots \\ \hline A_{M1} & \cdots & A_{MK} \end{array}\right), \text{ and} \quad (5)$$

$$B = \left(\begin{array}{c|c|c} B_{11} & \cdots & B_{1N} \\ \hline \vdots & \ddots & \vdots \\ \hline B_{K1} & \cdots & B_{KN} \end{array}\right) \quad (6)$$

where C_{ij} is $m_i \times n_j$, A_{ip} is $m_i \times k_p$, and B_{pj} is $k_p \times n_j$. (While this partitioning looks remarkably like the one in Section 3.3, the discussion in that section has no bearing on the discussion below.) Then C can be computed by a scaling $C \leftarrow \beta C$ followed by updates $C_{ij} \leftarrow \alpha A_{ip} B_{pj} + C_{ij}$, for $i = 1, \ldots, M$, $j = 1, \ldots, N$, $p = 1, \ldots, K$. Each of these individual updates can use the error detection schemes described above. Using this method workspace can be greatly reduced as can the cost of a recomputation. Moreover, there are a number of opportunities for the reuse of results $B_{pj}w$, $v^T A_{ip}$, $||B_{pj}||_\infty$, and $||A_{ip}||_\infty$, where w and v have length n_j and m_i, respectively.

Notice that the proposed error detection and correction scheme can now handle multiple errors with respect to the overall matrix-matrix multiplication, as long as only one error occurs during the computation $A_{ip}B_{pj}$.

5 An Actual Implementation

In this section we briefly outline our implementation of the ideas presented above.

We start by describing a high-performance implementation of matrix-matrix multiplication, ITXGEMM [8], developed at UT-Austin in collaboration with Dr. Greg M. Henry of the Intel Corporation. To understand how ITXGEMM uses hierarchical memory to attain high performance recall that the memory hierarchy of a modern microprocessor is often viewed as a pyramid (see Fig. 1). At the top of the pyramid there are the processor registers, with extremely fast access. At the bottom, there is disk and even slower media. As one goes down the pyramid, the amount of memory increases along with the time required to access that memory.

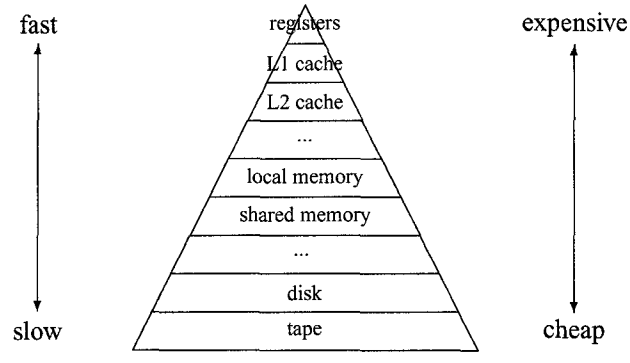

Figure 1. Hierarchical layers of memory.

As is well-known, processor speed has been increasing much faster than memory speed and it is thus memory bandwidth that limits the speed attained in practice for a given operation. Fortunately, matrix-matrix multiplication involves $2mnk$ FLOPs and only $mn + mk + kn$ data items. Thus, by carefully moving data between layers of memory, high-performance can be attained. *Note that the cost of error detection is of the same order as the cost for loading and storing to and from a memory layer.*

The particular implementation of matrix-matrix multiplication in ITXGEMM, which we modified as part of this re-

search, partitions C, A, and B as in (4)–(6). The partitioning scheme used for A is selected so that A_{ip} fills a large part of the L2 cache. For the architecture selected to act as a testbed, an Intel Pentium (R) III, the optimal partitioning turns out to be $m_i = k_p = 128$. (In other words, in the previous discussion A_{ip} is 128×128.) Then B is partitioned so that a reasonable amount of workspace is required for our right-sided error detection scheme. In particular, we chose $n_j = 512$. This means that the matrices are partitioned exactly as in (1)–(3) and updated as required by the left-sided error detection scheme, with $b = 8$. Code for error detection and correction was a straightforward addition to an implementation that naturally blocked for efficient utilization of the L1 and L2 caches of the Pentium (R) III processor.

If we consider all floating point operations to be equal and we count the cost of computing the norm of an $m \times n$ matrix as mn FLOPs, we expect the ratios of overhead to useful computation shown in Table 2. The overhead for correction is for the case when exactly one corruption occurs during the entire computation. This correction overhead scales linearly with the number of corruptions. The cost per FLOP of a matrix-vector multiplication is often an order of magnitude greater than the cost per FLOP of a matrix-matrix multiplication. Thus the above analysis for the cost of error detection may be optimistic by an order of magnitude. On the other hand, as mentioned, there are opportunities for amortizing the cost of the computation of matrix-vector multiplies and norms of matrices which are not taken into account in the analysis.

6 Experimental Results

All our experiments were performed on an Intel Pentium (R) III processor with a 650 MHz clockrate, 16 Kbytes of L1 data cache and 256 Kbytes of L2 cache, using IEEE double-precision floating point arithmetic ($\mathbf{u} \approx 2.2 \times 10^{-16}$). We report performance in MFLOPs/sec. (millions of floating point operations per second). Notice that the best performance we have seen on this particular processor with a high-performance matrix-matrix multiplication is around 530 MFLOPs/sec.

6.1 Fault-tolerance under simulated fault conditions

In order to evaluate the reliability of our error detection and correction techniques we decided to mirror in our experiments what we expect to be a realistic fault condition behavior in practice. Thus, instead of introducing an error either in A or B before the computation starts, we introduce the error before one of the updates of the form $C_{ij} \leftarrow \alpha A_{ip} B_{pj} + \beta C_{ij}$ is computed. The exact update, the entry were the error appeared (including the matrix, A or B), and its magnitude are randomly determined.

We do not analyze the case in which the error appears in C since, as stated in our theory (see Table 1), that error will always be detected using any of the detection methods, (at least, as long as it makes a non-negligible difference in the result).

The error detection mechanisms performed exactly as expected. All errors of significance that were introduced in matrix A were detected by the left-sided and two-sided detection methods. Similarly, all errors of significance that were introduced in matrix B were detected by the right-sided and two-sided detection methods. In practice both left- and right-sided methods detected errors of significance introduced in either A or B. As predicted, whenever we created a matrix A such that the elements in individual columns added to zero, the left-sided detection method had trouble detecting errors introduced in B. Whenever we created a matrix B such that the elements in individual rows added to zero, the right-sided detection method had trouble detecting errors introduced in A.

6.2 Performance evaluation

Next, we evaluated the overhead introduced in practice by our error detection/correction techniques. We added the error detection and correction mechanisms described in the previous sections to the implementation of matrix-matrix multiplication described in ITXGEMM. In [8, 9] we show that this implementation (without error detection and correction) is highly competitive with other efforts (e.g. [16], which does not address fault-tolerance) in providing high-performance matrix-matrix multiplication for the Intel Pentium (R) III processor.

We report results for the following fault-tolerant matrix-matrix multiplication implementations:

- L/R/2-sided detect: ITXGEMM-based implementation with left/right/two-sided detection.
- L/R/2-sided correct: ITXGEMM-based implementation with left/right/two-sided detection and correction.

Specifically, the error detection and and correction mechanisms were added to matrix-matrix multiplication algorithm `MPP-MMP-MPM` described in [8]. A significant error was introduced in matrix A as described in the previous subsection. The error was always detected and, if desired, corrected.

Figure 2 shows the performance achieved by the different matrix-matrix multiplication implementations for rank-k updates ($m = n$, $k = 128$) and general square matrix-matrix multiplication ($m = n = k$). For this prototype implementation, the error detection methods reduce performance by 20-25% even if no error is introduced. When a

Method	Overhead ($m_i = k_p = 128, n_j = 512, b = 8$)			
	$m = n = 512, k = 128$		$m = n = k = 512$	
	Detection	Correction	Detection	Correction
right-sided	2.2%	25%	2.2%	6%
left-sided	2.2%	0.4%	2.2%	0.1%
two-sided	4.4%	0.4%	4.4%	0.1%

Table 2. Theoretical overhead for error detection and correction.

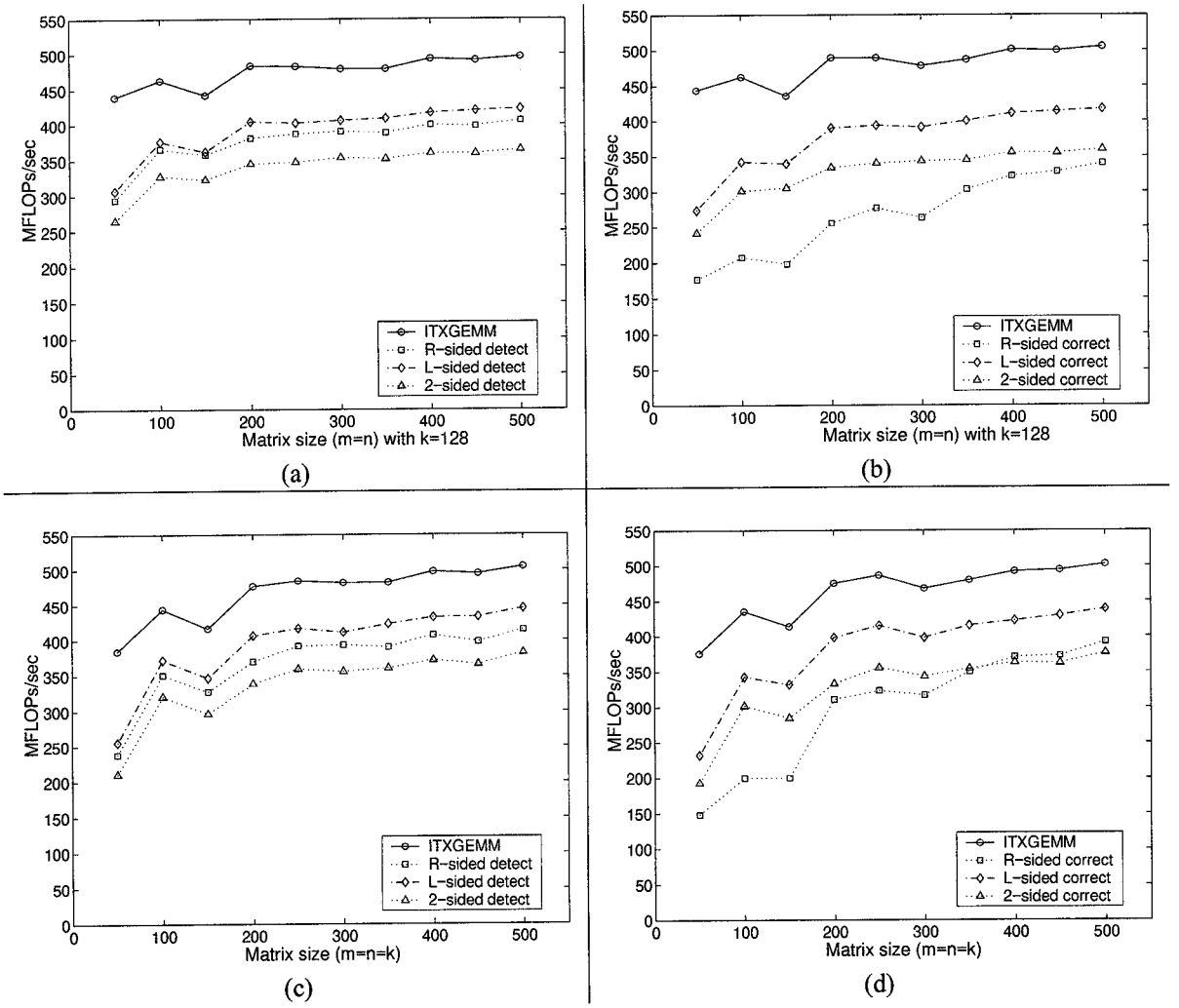

Figure 2. Performance of the matrix-matrix multiplication kernels.

single error is introduced and corrected, the performance of the right-sided detection and correction method is significantly worse. The performance of the left-sided method is not significantly affected. This supports the observations made in Section 4.2. Since the left-sided error detection and correction methods also detects and corrects virtually all errors introduced in B, the two-sided method is also not significantly affected.

It should be noted that we expect to reduce overhead significantly by carefully amortizing the required additional computations. Furthermore, while we currently tie the blocking used for the error detection and correction mechanisms to the blocking used by ITXGEMM for moving data into the L2 cache, overhead for the detection mechanism can be reduced if a coarser blocking were allowed. This would be at the expense of additional memory required for the roll-back mechanism as well as a higher computational overhead if a correction becomes necessary.

7 Status

We currently have a complete implementation of the above ideas for the operations

$$\begin{aligned} C &\leftarrow \alpha AB + \beta C \\ C &\leftarrow \alpha A^T B + \beta C \\ C &\leftarrow \alpha AB^T + \beta C \\ C &\leftarrow \alpha A^T B^T + \beta C \end{aligned}$$

Using similar techniques, we have also created fault-tolerant implementations for all the level 3 BLAS operations using our Formal Linear Algebra Methods Environment [9, 10]. While we currently only target double-precisions real arithmetic, and only have implementations for the Intel Pentium (R) III processor, the techniques are easily extended to single-precision or complex arithmetic and to other architectures.

The ultimate goal is to create an environment for developing fault-tolerant linear algebra libraries, the Formal Linear Algebra Recovery Environment (FLARE), which may eventually include fault-tolerant implementations for the major operations included in LAPACK.

8 Conclusion

In this paper, we have significantly extended the theory behind and practice of algorithmic fault-tolerant matrix-matrix multiplication. In particular, we have expanded upon existing results relevant to the detection of errors in the computation $C = AB$. Based on these theoretical results, we have provided a practical, fault-tolerant, high-performance implementation of the matrix-matrix multiplication operation. It should be obvious that the results extend to all cases of matrix-matrix multiplication that are part of the BLAS. The experimental results demonstrate that our methods introduce, in practice, an acceptable level of overhead (about 20% for the error detection mechanism and an insignificant additional amount when a correction is required) relative to high-performance implementations that do not include algorithmic fault-tolerance.

Additional Information

For additional information:

`www.cs.utexas.edu/users/flame/FLARE/`.

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